Representations of the Lorentz Group Corresponding to Unstable Particles

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The irreducible nonunitary representations of the inhomogeneous Lorentz group which satisfy the condition that the complex energy-momentum 4 vectors be transformable into a rest frame by a real Lorentz rotation are found. By explicit construction of wave packets in configuration space it is seen that the vectors of the representation space have a natural interpretation as states of unstable particles. The new representations arise in the study of complex singularities associated with resonance poles in analytically continued scattering amplitudes.

PREVIOUS studies[1,2] of the properties of analytically continued scattering amplitudes have shown that a particle of complex mass, identified as a pole of the amplitude at a complex point, shares many of the formal properties of a true stable particle. We would like to show that such a particle corresponds to the physical system consisting of an unstable particle and that such a system has a Lorentz-invariant characterization, even though it does not satisfy the most common criterion for a particle, namely, it does not correspond to an irreducible unitary representation of the inhomogeneous Lorentz group.[3] However, the unitarity of the representation corresponds to the conservation of probability. Since we are interested in particles which decay, it is natural to turn to nonunitary representations.

At present, the numerous irreducible nonunitary representations already known[4]—which correspond to real energy-momentum 4 vectors and real square mass —have no obvious physical interpretation. If, in addition, we allow all nonunitary representations with arbitrary complex 4 vectors, we open a Pandora's box and let out a large assortment of physically obscure representations. The problem is to find an additional requirement so that only those with a physical interpretation are obtained. This problem is present, though in less acute form, even for unitary representations, since not all of these have a physical interpretation.

The representations of the inhomogeneous Lorentz group may be obtained by first finding representations of the normal subgroup of space-time translations $x \to x+a$, where a is a real 4 vector. Since it is an Abelian group, its irreducible representations are

$$|p,\alpha\rangle \to \exp(-ip\cdot a)\;|p,\alpha\rangle\,, \tag{1}$$

where p is an arbitrary complex 4 vector and α is a label for later use. The invariant of the full group $p^2=s_r$ is an arbitrary complex number which characterizes the representation. It follows from the multiplication law of the group elements that under an arbitrary real Lorentz rotation $x\to \Lambda x$ the eigenvectors of translation transform as

$$|p,\alpha\rangle \to \textstyle\sum_\beta \mathfrak{D}_{\beta\alpha}(p,\Lambda)\,|\Lambda p,\beta\rangle\,. \tag{2}$$

It is perhaps worth emphasizing that, although p is complex, we are nevertheless looking at representations of the *real* inhomogeneous Lorentz group, i.e., the displacement vectors a and rotation matrices Λ are real. We exclude space and time inversions.

So far, our treatment is general. At this point we propose the requirement that the 4-vector p be transformable by a real Lorentz rotation Λ into a rest system, where it has the form

$$s_r^{1/2}(1,0,0,0)\,, \tag{3}$$

where s_r is a complex number. (We ignore the problem of unstable particles of zero mass and take $s_r\neq 0$.) Consequently, every 4-vector p of the representation s_r may be obtained by the inverse Lorentz transformation and thus is of the form $p=s_r^{1/2}((\mathbf{u}^2+1)^{1/2},\,\mathbf{u})$, where $\mathbf{u}$ is a real 3 vector related to the velocity $\mathbf{v}$ of the Lorentz transformation by $\mathbf{u}=(1-\mathbf{v}^2)^{-1/2}\mathbf{v}$.

For 4-vectors which may be brought into the form (3), the "little group" [i.e., the subgroup of Lorentz rotations which leaves the form (3) invariant] is the three-dimensional rotation group. Its irreducible representations are all of finite dimension corresponding to integral or half-integral spin j, the second invariant of the full group. We thus characterize unstable particles by square mass s_r and spin j, and associate states of such a particle with the vectors of the representation space. They may be labelled by $|s_r,\mathbf{u},j,\lambda\rangle$ corresponding to 4-momentum $p=(s_r)^{1/2}((\mathbf{u}^2+1)^{1/2},\,\mathbf{u})$, spin j, and helicity λ.

At this point we may make a few observations. (1) In the nonunitary representations we have obtained, it is the space-time translations that correspond to nonunitary operators, whereas the operators corresponding to Lorentz rotations are unitary. (2) The tensor product of two of the representations obtained here cannot in general be reduced to a tensor sum of such representations, in contrast to the case of real mass. (3) The representations obtained here correspond to momentum

[1] H. Stapp, Lawrence Radiation Laboratory Report No. 10261 (unpublished). I am grateful to Dr. Stapp for making this work available to me before publication.
[2] D. Zwanziger, Phys. Rev. **131**, 888 (1963).
[3] E. P. Wigner, Ann. Math. **40**, 149 (1939). The method and terminology of this and the following reference has been adopted.
[4] Iu. Shirokov, Zh. Eksperim. i Teor. Fiz. **33**, 861 (1957), **33**, 1196 (1957), and **33**, 1208 (1957) [translation: Soviet Phys.—JETP **6**, 664 (1958), **6**, 919 (1958), and **6**, 929 (1958)].

4-vectors of the form λp, where λ is a complex number and p a real time-like 4-vector. It is clear that more new representations can easily be obtained from the already known representations[3,4] by considering momentum vectors of the form λp, where p is any real 4-vector.

It is seen that with the requirement (3) we obtain states only slightly more general than those of Wigner. This requirement is admittedly *ad hoc*, but is equivalent to the dynamically reasonable assumption that resonance poles appear in only one partial wave in an angular momentum decomposition of the amplitude. This assumption means that the "little group" is the rotation group, which in turn means that the energy-momentum 4 vector may be brought into the form (3).

For a space-time interpretation we construct wave packets of the free particle states by Fourier transform. We set $(s_r)^{1/2}=m-i\gamma$. Because of the sign ambiguity in the square root we may choose $m>0$. Poles occur in complex conjugate pairs (reality condition of the amplitude) and we also choose $\gamma>0$ for consideration. The wave packet may then be expressed

$$\psi(\mathbf{x},t)=\int\varphi(\mathbf{u})\exp\{-i(m-i\gamma)\times[(\mathbf{u}^2+1)^{1/2}t-\mathbf{u}\cdot\mathbf{x}]\}d\mathbf{u}. \quad (4)$$

In the usual way we assume that $\varphi(\mathbf{u})$ is very sharply peaked around $\mathbf{u}=\mathbf{u}_0$, so that

$$i\partial_\mu\psi\approx(m-i\gamma)((\mathbf{u}_0^2+1)^{1/2},\,\mathbf{u}_0)\psi\equiv p_\mu\psi\,, \quad (5)$$

and expand the exponential up to terms linear in $\mathbf{u}-\mathbf{u}_0$ thereby neglecting spreading of the wave packet:

$$\psi(\mathbf{x},t)=\exp\{-i(m-i\gamma)[(\mathbf{u}_0^2+1)^{1/2}t-\mathbf{u}_0\cdot\mathbf{x}]\}\times\int\varphi(\mathbf{u})\exp\{-i(m-i\gamma)(\mathbf{u}-\mathbf{u}_0)\cdot[(\mathbf{u}_0^2+1)^{-1/2}\mathbf{u}_0t-\mathbf{x}]\}d\mathbf{u}\,, \quad (6)$$

or

$$\psi(\mathbf{x},t)=\exp[-i(m-i\gamma)(\mathbf{u}_0^2+1)^{-1/2}t]\times A(\mathbf{x}-(\mathbf{u}_0^2+1)^{-1/2}\mathbf{u}_0t)\,, \quad (7)$$

where

$$A(\mathbf{r})=\exp[i(m-i\gamma)\mathbf{u}_0\cdot\mathbf{r}]\times\int\varphi(\mathbf{u})\exp[i(m-i\gamma)(\mathbf{u}-\mathbf{u}_0)\cdot\mathbf{r}]d\mathbf{u}.$$

We may now form the four current

$$j_\mu=\tfrac{1}{2}i(\psi^*\partial_\mu\psi-\partial_\mu\psi^*\psi)\approx\tfrac{1}{2}(p_\mu+p_\mu^*)\psi^*\psi\,, \quad (8)$$

where we have made use of Eq. (5).

From Eq. (7) we have

$$j_\mu=m(1-v^2)^{-1/2}(1,\mathbf{v})\exp[-2\gamma(1-v^2)^{1/2}t]\times|A(\mathbf{x}-\mathbf{v}t)|^2, \quad (9)$$

$$j_\mu=m(1-v^2)^{-1/2}(1,\mathbf{v})\exp[-2\gamma v^{-1}(1-v^2)^{1/2}\hat{v}\cdot\mathbf{x}]\times|A'(\mathbf{x}-\mathbf{v}t)|^2, \quad (10)$$

where

$$\mathbf{v}=(\mathbf{u}_0^2+1)^{-1/2}\mathbf{u}_0$$

and

$$|A'(\mathbf{r})|^2=\exp(2\gamma u_0^{-1}\hat{u}_0\cdot\mathbf{r})|A(\mathbf{r})|^2.$$

These expressions are physically transparent, corresponding to a wave packet travelling with a real velocity $\mathbf{v}$, $|\mathbf{v}|<1$, decaying exponentially with a decay time $\tau=[2\gamma(1-v^2)^{1/2}]^{-1}$ that transforms relativistically like a time interval, decay length $v\tau$, and particle current proportional to the momentum 4 vector formed with the velocity $\mathbf{v}$ and real mass m.

A possible use of the new representations is in the direct construction of S-matrix elements with unstable external particles according to the method of Stapp,[5] and Barut, Muzinich, and Williams.[6] Such elements have been previously defined,[1,2] by factorization of the residue at complex poles. It was found that other discontinuities of the amplitude may be expressed as a sum over states including those of unstable particles. The corresponding phase space (invariant measure) was found in a special case, Eq. (27) of Ref. 2, by explicit calculation. In general, it may be obtained from the phase space for n stable particles, expressed as a finite multiple integral over a finite region, by analytic continuation in the n square masses m_i^2.

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[5] H. Stapp, Phys. Rev. **125**, 2139 (1962).

[6] A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).